

Anticanonical Iitaka dimensions in contractions in characteristic $p > 0$

X normal proj var. / $k = \bar{k}$, $K_X =$ canonical divisor joint w/ Jacopo Brivio & Chi-Kang Cheng

A measure of positivity of K_X is the Kodaira dimension

L line bundle on X , its Iitaka dimension $\kappa(X, L)$ is:

$$\varphi_{|mL|} : X \dashrightarrow \mathbb{P}^N$$

$$\kappa(X, L) := \begin{cases} -\infty & \text{if } |mL| = \emptyset \ \forall m \\ \dim \text{im}(\varphi_{|mL|}) & \text{for } m \gg 0 \text{ otherwise} \end{cases}$$

when $L = K_X$, $\kappa(X, K_X) =$ Kodaira dim.

Q If $f: X \rightarrow Y$ contraction ($f_* \mathcal{O}_X = \mathcal{O}_Y$), how do $\kappa(X, K_X)$, $\kappa(\Phi, K_\Phi)$, $\kappa(Y, K_Y)$ relate?
 \uparrow general fibre

easy additivity thm: $\kappa(X, L) \leq \kappa(\Phi, L|_\Phi) + \dim(Y)$

Iitaka conjecture: $\kappa(X, K_X) \geq \kappa(\Phi, K_\Phi) + \kappa(Y, K_Y)$

Q What about $\kappa(X, -K_X)$?

Cn,m thm [Chang '22]

$f: X \rightarrow Y$ contraction / $k = \mathbb{Z}$ of char. 0

X, Y smooth proj. var.

$-K_X$ \mathbb{Q} -eff and $B(-K_X)$ does not dominate Y

"
 $\cap B_S(-mK_X)$

$\Rightarrow \kappa(X, -K_X) \leq \kappa(\Phi, -K_\Phi) + \kappa(Y, -K_Y)$

Sketch of pf

Step 1 find $\Delta \geq 0$ \mathbb{Q} -div on X s.t. $\begin{cases} K_X + \Delta \sim_{\mathbb{Q}} 0 \\ (\Phi, \Delta_\Phi) \text{ has nice singul.} \end{cases}$

Step 2 apply a canonical bundle formula
 $\exists \Delta_Y \geq 0$ s.t. $K_X + \Delta \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y)$

Step 3 prove the inequality when $\kappa(Y, -K_Y) = 0$ $\left(\begin{array}{l} -K_Y \text{ is } \mathbb{Q}\text{-eff.} \\ \text{enough to show } \alpha: H^0(X, -mK_X) \xrightarrow{\text{res}} H^0(\Phi, -mK_\Phi) \end{array} \right)$
is inj.

Assume Y curve

$\exists ! \alpha \in H^0(-K_Y)$, $y \in Y \notin \text{supp}(M)$, $\Phi = f^{-1}(y)$
If α not inj. $\Rightarrow \exists 0 \leq N \sim_{\mathbb{Q}} -K_X$ $\Phi \subseteq \text{supp}(N)$

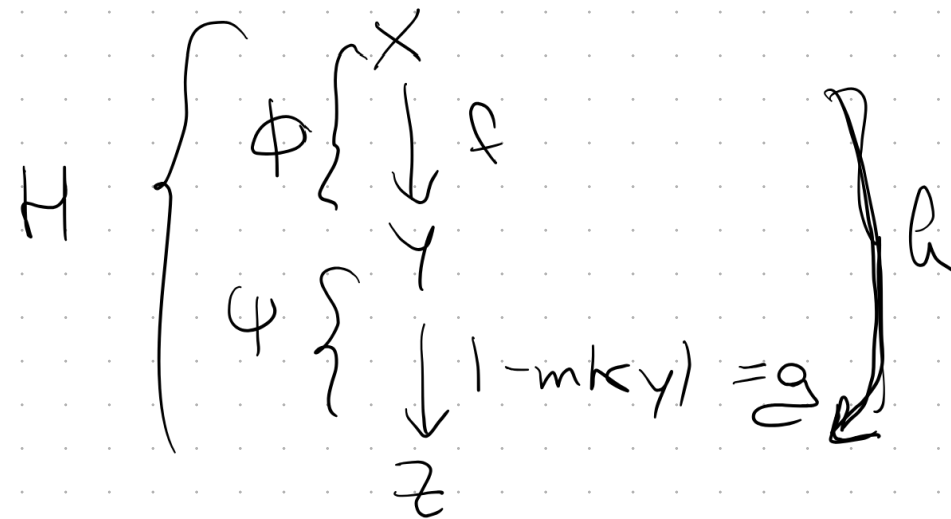
$$\beta = \text{coeff}_\Phi(N), \quad \sum_Q \beta \Phi \geq 0$$

$$-k_X - \beta \Phi$$

$$\Rightarrow -k_Y - \varepsilon \beta y \text{ is } \mathbb{Q}\text{-eff } \alpha \varepsilon \ll 1$$

$$y \Phi \approx \sum_Q -k_Y \approx \mathbb{Q} \leq 0 \quad \underbrace{\varepsilon \beta y}_{\approx 0} \quad \approx$$

Step 4 reduce to $\kappa(Y, -k_Y) = 0$



$$\kappa(Y, -k_Y) = 0$$

$$\boxed{H \rightarrow \Psi} \Rightarrow \kappa(H, -k_H) \leq \kappa(\Phi, -k_\Phi)$$

easy additivity on \mathbb{R} : $\kappa(X, -k_X) \leq \kappa(H, -k_H) + \dim Z$
 $\leq \kappa(\Phi, -k_\Phi) + \kappa(Y, -k_Y)$

✓

F-singularities

$$\text{char } k = p > 0$$

Frobenius morphism

$$F^e: X \rightarrow X \quad \mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X$$
$$x \mapsto x \quad s \mapsto s^{p^e}$$

X is globally F-split (GFS) if

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X \quad \text{admits a splitting for } e \gg 0$$
$$\begin{array}{ccc} & & \downarrow \\ \text{id} \searrow & & \mathcal{O}_X \end{array}$$

X is globally F-regular (GFR) if
 $\forall D \geq 0$ there, $\exists e \gg 0$ s.t.

$$\mathcal{O}_X \rightarrow F_*^e \mathcal{O}_X(D) \quad \text{splits}$$
$$\begin{array}{ccc} & & \downarrow \\ \text{id} \searrow & & \mathcal{O}_X \end{array}$$

(not) ex. [Kunz] X regular iff $F_* \mathcal{O}_X$ is locally free

↳ not always GFS/GFR

E elliptic curve is GFS iff it's ordinary

$$E[p] = \begin{cases} 0 & \text{SUPER SING.} \\ \mathbb{Z}/p\mathbb{Z} & \text{ORDINARY} \end{cases}$$

Rank [Schwede-Smith]

- ① If X is GFS $\Rightarrow \exists \Delta \geq 0$ s.t. (X, Δ) is GFS and $K_X + \Delta \sim_{\mathbb{Q}} 0$
- ② If X GFR $\Rightarrow \exists \Delta \geq 0$ s.t. (X, Δ) is GFR and $-(K_X + \Delta)$ is ample

Canonical bundle formula

there is a correspondence between:

$$f^* L \xrightarrow{\psi \neq 0} \mathcal{O}_X \quad \text{and} \quad \Delta \geq 0 \text{ s.t. } (1 - pe)(K_X + \Delta) \text{ Cartier}$$

L line bundle

$$(L = \mathcal{O}_X((1 - pe)(K_X + \Delta)))$$

thm.

① [Das-Schwede] $f: X \rightarrow Y$ contraction, $\exists e$ s.t. $(1 - pe)K_X \sim f^* L$

and X_{η} generic fibre is GFS

$$\Rightarrow \exists \Delta_Y \geq 0 \text{ s.t. } K_X \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y)$$

$$\text{If } X \text{ is GFS} \Rightarrow (Y, \Delta_Y) \text{ GFS}$$

② $f: X \rightarrow Y$ finite, $p \mid \deg(f)$, X is GFS and

$$\exists e \text{ s.t. } (1 - pe)K_X \sim f^* L$$

$$\Rightarrow \exists \Delta_Y \geq 0 \text{ s.t. } K_X \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y) \text{ and } (Y, \Delta_Y) \text{ GFS}$$

Sketch of (18)

$$F_*^e \mathcal{O}_X((1-p^e)K_X) \xrightarrow{q \neq 0} \mathcal{O}_X$$

$f_* \mathcal{L}$

$$F_*^e \mathcal{O}_Y(L) \otimes F_*^e f_* \mathcal{O}_Y \rightarrow f_* \mathcal{O}_Y$$

$\downarrow \quad \text{Tr}(f)/\deg(f) \quad \downarrow$

$$F_*^e \mathcal{O}_Y(L) \xrightarrow{f_* \varphi \neq 0} \mathcal{O}_Y \quad f_* \varphi \neq 0$$

because X_Y is GFS

$$\Rightarrow \exists \Delta_Y \geq 0 \text{ s.t. } f_*(K_Y + \Delta_Y) \cong \mathcal{O}_X$$

$C_{n,m}$

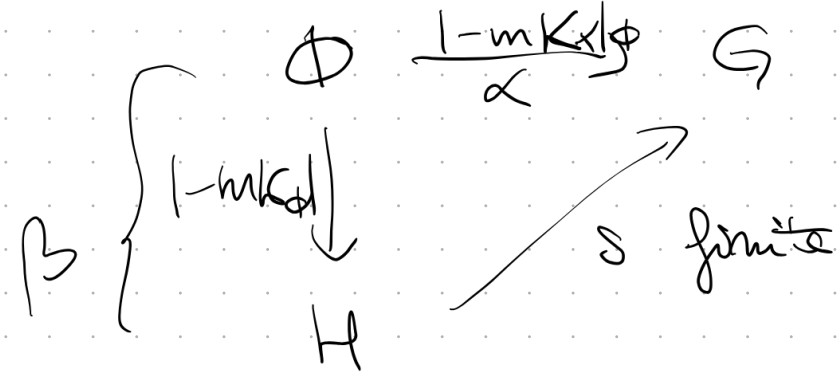
thm [B., Brivio, Chang '23] $f: X \rightarrow Y$ of smooth proj var.

($k = \mathbb{F}_2$ of char. $p > 0$) s.t. a general fibre Φ is k -globally F -regular

Assume $\exists m \in \mathbb{N} \setminus p\mathbb{N}$ s.t. $-mK_X$ Cartier and $|mK_X|_\Phi$ induces a map $\mathbb{A}^1/\mathbb{G}_m$ of degree not divisible by p

$$\Rightarrow \kappa(X, -K_X) \leq \kappa(\Phi, -K_\Phi) + \kappa(Y, -K_Y)$$

- $| -mK_X |_\Phi$ BPF
- \wedge
- $| -mK_\Phi |$ BPF



$P \setminus \text{deg}(S)$

K -glob. F-Reg. \therefore

Φ_η is GFS and

(H, Δ_H) induced pair is GFR

RMK

$$\beta^*(K_H + \Delta_H) \sim_{\mathbb{Q}} K_\Phi$$

\Rightarrow (G, Δ_G) GFR and
CBF

$$H^0(X, -mK_X) \otimes_{\mathbb{C}} \cong H^0(\Phi, -m\alpha^*(K_G + \Delta_G))$$

$\Rightarrow \exists \Gamma_G \geq 0$ and s.t. $K_G + \Delta_G + \Gamma_G \sim_{\mathbb{Q}} 0$

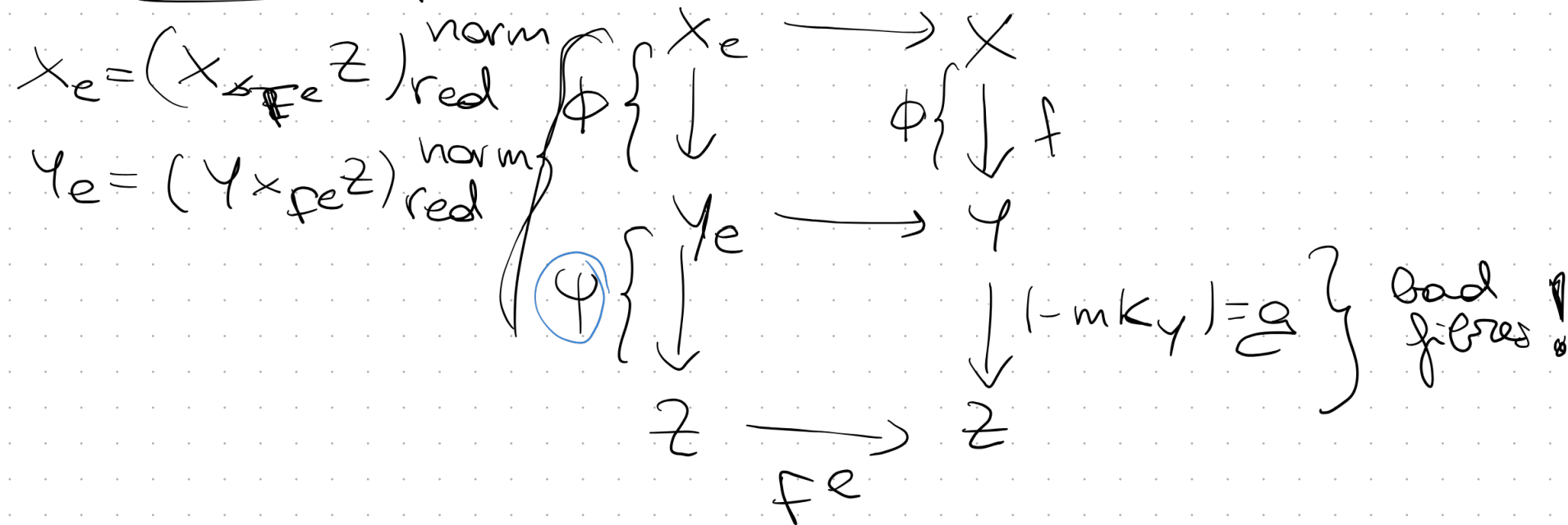
[Schwede-Smith]

and $(G, \Delta_G + \Gamma_G)$ is GFS

we found $\Gamma \geq 0$ on $X \iff \alpha^* \Gamma_G$

and $K_X + \Gamma \sim_{\mathbb{Q}} 0$
 (Φ, Γ_Φ) is GFS

About Step 4



for $e \gg 0$, $Y_e \rightarrow Z$ has reduced normal fibres

\Rightarrow apply "step 4" to $X_e \rightarrow Y_e \rightarrow Z$

[Fatakhali-Waldron]: relate K_X and K_{X_e}
 K_Y and K_{Y_e}

\Rightarrow we can conclude

Itaka Conjecture

State of the art:

Char. 0

- Viehweg '77, '82, '83
- Kawamata '81, '82, '85
- Fujino '03
- Birkar '09
- Chen, Hacon '11
- Cao Păun '17

- Hacon, Popa, Schnell
- Cao '18 '17
- ...
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In particular,
proven if
 $\dim Y \leq 2$ and if
 $X_{\bar{y}}$ admits a good
minimal model

Char. $p > 0$

- 📖 Cascini, Ejiri, Kollár,
- Zhang '19
- Chen, Zhang '15
- Ejiri '17
- Birkar, Chen, Zhang '18
- Ejiri, Zhang '18

} counterexamples

In particular, proven if
 $\dim X \leq 3, p > 5$ and if
 f generically smooth of
relative dimension 1.

Outline of the proof

- transfer positivity from $-K_X$ to $-K_Y$
- ① find a complement $\Delta \geq 0$ s.t. $\left. \begin{array}{l} K_X + \Delta \sim_{\mathbb{Q}} 0 \\ (\Phi, \Delta_{\Phi}) \text{ has nice singularities} \end{array} \right\}$ (GFS)
 - ② apply some canonical bundle formula
 - ③ prove the inequality when $\kappa(Y, -K_Y) = 0$
 - ④ reduce to the case $\kappa(Y, -K_Y) = 0$
- Q And in char. $p > 0$? ☺ qed