Anticanonical Iitaka dimensions in contractions in characteristic $p>0$

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X normal proj. var. $/k=\bar{k}$, K_X = canonical divisor

A measure of positivity of K_X is the Kodaira dimension

L line bundle on X, its Iitaka dimension $\kappa(X,L)$ is:

$$\varphi_{|mL|}: X \dashrightarrow \mathbb{P}^N$$

$$\kappa(X,L) := \begin{cases} -\infty & \text{if } |mL| = \emptyset \ \forall m \\ \dim \operatorname{im}(\varphi_{|mL|}) & \text{for } m \gg 0 \text{ otherwise} \end{cases}$$

when $L=K_X$, $\kappa(X,K_X)$ = Kodaira dim.

<u>Q</u> If $f: X \to Y$ contraction ($f_* \mathcal{O}_X = \mathcal{O}_Y$), how do $\kappa(X,K_X)$, $\kappa(\Phi,K_\Phi)$, $\kappa(Y,K_Y)$ relate?

general fibre

easy additivity thm: $\kappa(X,L) \leq \kappa(\Phi, L|_\Phi) + \dim(Y)$

Iitaka conjecture: $\kappa(X,K_X) \geq \kappa(\Phi,K_\Phi) + \kappa(Y,K_Y)$

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Q What about $\kappa(X, -K_X)$?

$C_{n,m}$ thm [Chang '22]

$f: X \to Y$ contraction / $k = \bar{k}$ of char. 0

X, Y smooth proj. var.

$-K_X$ $\mathbb{Q}$-eff and $\mathbb{B}(-K_X)$ does not dominate Y

$\mathbb{B}(-K_X) = \bigcap_{m \in \mathbb{N}} Bs(-mK_X)$

$\Rightarrow \kappa(X, -K_X) \leq \kappa(\Phi, -K_\Phi) + \kappa(Y, -K_Y)$

Sketch of pf

Step 1 find $\Delta \geq 0$ $\mathbb{Q}$-div. on X s.t. $\begin{cases} K_X + \Delta \sim_{\mathbb{Q}} 0 \\ (\Phi, \Delta_\Phi) \text{ has nice singul.} \end{cases}$

Step 2 apply a canonical bundle formula

$\exists\, \Delta_Y \geq 0$ s.t. $K_X + \Delta \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} f^*(K_Y + \Delta_Y)$

$-K_Y$ is $\mathbb{Q}$-eff.

Step 3 prove the inequality when $\kappa(Y, -K_Y) = 0$

enough to show $\alpha: H^0(X, -mK_X) \xrightarrow{res} H^0(\Phi, -mK_\Phi)$ is inj.

Assume Y curve

$\exists!\, 0 \leq M \sim_{\mathbb{Q}} -K_Y$, $y \in Y$, $y \notin Supp(M)$, $\Phi = f^{-1}(y)$

If α not inj. $\Rightarrow \exists\, 0 \leq N \sim_{\mathbb{Q}} -K_X$ $\Phi \subseteq Supp(N)$

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$$\beta = \operatorname{coeff}_{\Phi}(N), \quad N - \beta\Phi \geq_{\mathbb{Q}} 0$$

$$-K_X - \beta\Phi$$

$$\Longrightarrow -K_Y - \varepsilon\beta y \text{ is } \mathbb{Q}\text{-eff } 0 < \varepsilon \ll 1$$

$$\varphi_* M \sim_{\mathbb{Q}} -K_Y \sim_{\mathbb{Q}} \underset{\geq 0}{\Gamma} + \underline{\varepsilon\beta y} \quad ⚡$$

<u>Step 4</u> reduce to $\kappa(Y, -K_Y) = 0$

$$\begin{tikzcd}
X \arrow[d, "f"] \arrow[dd, bend left, "h"] \\
Y \arrow[d, "|-mK_Y| = g"] \\
Z
\end{tikzcd}$$

H, Φ, Ψ

$$\kappa(\Psi, -K_\Psi) = 0$$

$$\boxed{\underbrace{H \longrightarrow \Psi}_{\Phi}} \Longrightarrow \kappa(H, -K_H) \leq \kappa(\Phi, -K_\Phi)$$

easy additivity on h: $\kappa(X, -K_X) \leq \kappa(H, -K_H) + \dim Z$

$$\leq \kappa(\Phi, -K_\Phi) + \kappa(Y, -K_Y)$$

✓

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F-singularities

char $k = p > 0$

Frobenius morphism

$F^e : X \to X$ $\qquad$ $\mathcal{O}_X \to F^e_* \mathcal{O}_X$

$x \mapsto x$ $\qquad$ $s \mapsto s^{p^e}$

X is globally F-split (GFS) if

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[dr, bend right, "id"'] & F^e_* \mathcal{O}_X \arrow[d] \\ & \mathcal{O}_X \end{tikzcd}$$

admits a splitting for $e \gg 0$

X is globally F-regular (GFR) if

$\forall D \geq 0$ Weil, $\exists e \gg 0$ s.t.

$$\begin{tikzcd} \mathcal{O}_X \arrow[r] \arrow[dr, bend right, "id"'] & F^e_* \mathcal{O}_X(D) \arrow[d] \\ & \mathcal{O}_X \end{tikzcd}$$

splits

(not) exa. [Kunz] X regular iff $F_* \mathcal{O}_X$ is locally free

↳ not always GFS / GFR

E elliptic curve is GFS iff it's ordinary

$$E[p] = \begin{cases} 0 & \text{SUPERSING.} \\ \mathbb{Z}/p\mathbb{Z} & \text{ORDINARY} \end{cases}$$

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Rmk [Schwede-Smith]

① If X is GFS $\Rightarrow \exists \Delta \geq 0$ s.t. (X,Δ) is GFS and $K_X+\Delta \sim_{\mathbb{Q}} 0$

② If X GFR $\Rightarrow \exists \Delta \geq 0$ s.t. (X,Δ) is GFR and $-(K_X+\Delta)$ is ample

Canonical bundle formula

there is a correspondence between:

$F^e_* \mathcal{L} \xrightarrow{\varphi \neq 0} \mathcal{O}_X$ and $\Delta \geq 0$ s.t. $(1-p^e)(K_X+\Delta)$ Cartier

$\mathcal{L}$ line bundle

$$\left(\mathcal{L} = \mathcal{O}_X((1-p^e)(K_X+\Delta))\right)$$

Thm.

① [Das-Schwede] $f: X \to Y$ contraction, $\exists e$ s.t. $(1-p^e)K_X \sim f^*L$ and X_η generic fibre is GFS

$\Rightarrow \exists \Delta_Y \geq 0$ s.t. $K_X \sim_{\mathbb{Q}} f^*(K_Y+\Delta_Y)$

If X is GFS $\Rightarrow (Y,\Delta_Y)$ GFS

② $f: X \to Y$ finite, $p \nmid \deg(f)$, X is GFS and

$\exists e$ s.t. $(1-p^e)K_X \sim f^*L$

$\Rightarrow \exists \Delta_Y \geq 0$ s.t. $K_X \sim_{\mathbb{Q}} f^*(K_Y+\Delta_Y)$ and (Y,Δ_Y) GFS

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Sketch of pf)

$$F^e_* \mathcal{O}_X(\underbrace{(1-p^e)K_X}_{f^*L}) \xrightarrow{\varphi \neq 0} \mathcal{O}_X$$

$$\Big\downarrow f_*$$

$$\begin{array}{ccc} F^e_*\mathcal{O}_Y(L) \otimes F^e_* f_* \mathcal{O}_Y & \longrightarrow & f_*\mathcal{O}_Y \\ \downarrow & Tr(f)/\deg(f) & \downarrow \\ F^e_* \mathcal{O}_Y(L) & \xrightarrow{f_*\varphi \neq 0} & \mathcal{O}_Y \end{array}$$

$f_*\varphi \neq 0$ because X_η is GFS

$\Rightarrow \exists \Delta_Y \geq 0$ s.t. $f^*(K_Y+\Delta_Y) \sim_{\mathbb{Q}} K_X$

$\overline{C_{n,m}}$

thm [B., Brivio, Chang '23] $f: X \to Y$ of smooth proj var.
/ $k=\bar{k}$ of char. $p>0$ s.t. a general fibre Φ is K-globally F-regular

Assume $\exists\, m \in \mathbb{N} \setminus p\mathbb{N}$ s.t. $-mK_X$ Cartier and $|-mK_X|_\Phi$ induces a map w/ St. degree not divisible by p

$$\Rightarrow \kappa(X,-K_X) \leq \kappa(\Phi,-K_\Phi) + \kappa(Y,-K_Y)$$

- $|-mK_X|_\Phi$ BPF
  $\cap|$
  $|-mK_\Phi|$ BPF

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$$\begin{tikzcd} \Phi \arrow[r, "|-mK_X|_\Phi"', "\alpha"] \arrow[d, "|-mK_\Phi|"', "\beta"] & G \\ H \arrow[ur, "S \text{ finite}"'] & \end{tikzcd}$$

$p \nmid \deg(S)$

K-glob. F-reg. : Φ_η is GFS and (H, Δ_H) induced pair is GFR

RMK $\quad \beta^*(K_H+\Delta_H) \sim_{\mathbb{Q}} K_\Phi$

$\underset{\text{CBF}}{\Longrightarrow}$ (G, Δ_G) GFR and

$\boxed{H^0(X, -mK_X)|_\Phi} \simeq H^0(\Phi, -m\alpha^*(K_G+\Delta_G))$

$\Longrightarrow$ $\exists\, \Gamma_G \geq 0$ and s.t. $K_G+\Delta_G+\Gamma_G \sim_{\mathbb{Q}} 0$

[Schwede-Smith] and $(G, \Delta_G+\Gamma_G)$ is GFS

we found $\underset{\text{on } X}{\Gamma} \overset{\geq 0}{\longleftrightarrow} \alpha^*\Gamma_G$ and $K_X+\Gamma \sim_{\mathbb{Q}} 0$

(Φ, Γ_Φ) is GFS

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About Step 4

$X_e = (X \times_{F^e} Z)^{\text{norm}}_{\text{red}}$

$Y_e = (Y \times_{F^e} Z)^{\text{norm}}_{\text{red}}$

$$\begin{tikzcd}
X_e \arrow[r] \arrow[d, "\phi"'] \arrow[dd, bend right, "\psi"'] & X \arrow[d, "\phi"', "f"] \\
Y_e \arrow[r] \arrow[d, "\psi"'] & Y \arrow[d, "|-mK_Y| = \emptyset"] \\
Z \arrow[r, "F^e"'] & Z
\end{tikzcd}$$

$\left.\right\}$ bad fibres!

for $e \gg 0$, $Y_e \to Z$ has reduced normal fibres

$\Rightarrow$ apply "Step 4" to $X_e \to Y_e \to Z$

[Patakfalvi–Waldron]: relate K_X and K_{X_e}
K_Y and K_{Y_e}

$\Rightarrow$ we can conclude

Iitaka conjecture

State of the art:

Char. 0

- Viehweg '77, '82, '83
- Kawamata '81, '82, '85
- Fujino '03
- Birkar '09
- Chen, Hacon '11
- Cao Paun '17

- Hacon, Popa, Schnell
- Cao '18 '17
- . . .

In particular, proven if $\dim Y \leq 2$ and if $X_{\bar{\eta}}$ admits a good minimal model

Char. $p>0$

- Cascini, Ejiri, Kollár, Zhang '19

counterexamples

- Chen, Zhang '15
- Ejiri '17
- Birkar, Chen, Zhang '18
- Ejiri, Zhang '18

In particular, proven if $\dim X \leq 3$, $p > 5$ and if f generically smooth of relative dimension 1.

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Outline of the proof

transfer positivity from $-K_X$ to $-K_Y$

1. find a complement $\Delta \geq 0$ s.t. $\begin{cases} K_X + \Delta \sim_{\mathbb{Q}} 0 \\ (\phi, \Delta_\phi) \text{ has nice singularities} \end{cases}$ GFS
2. apply some canonical bundle formula
3. prove the inequality when $\kappa(Y, -K_Y) = 0$
4. reduce to the case $\kappa(Y, -K_Y) = 0$

qed

Q And in char. $p > 0$?

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